

Preprint typeset in JHEP style - HYPER VERSION

TAUP-2835-06  
ULB-TH/06-20  
hep-th/0609194

# On the pp-wave limit and the BMN structure of new Sasaki-Einstein spaces

---

**Stanislav Kuperstein**

*Physique Théorique et Mathématique,  
Université Libre de Bruxelles and International Solvay Institutes,  
ULB-Campus Plaine C.P.231, B-1050 Bruxelles, Belgium.  
E-mail: [skuperst@ulb.ac.be](mailto:skuperst@ulb.ac.be)*

**Oded Mintkevich, Jacob Sonnenschein**

*School of Physics and Astronomy,  
The Raymond and Beverly Sackler Faculty of Exact Sciences,  
Tel Aviv University, Ramat Aviv, 69978, Israel.  
E-mail: [oded@post.tau.ac.il](mailto:oded@post.tau.ac.il), [cobi@post.tau.ac.il](mailto:cobi@post.tau.ac.il)*

**ABSTRACT:** We construct the  $pp$ -wave string associated with the Penrose limit of  $Y^{p,q}$  and  $L^{p,q,r}$  families of Sasaki-Einstein geometries. We identify in the dual quiver gauge theories the chiral and the non-chiral operators that correspond to the ground state and the first excited states. We present an explicit identification in a prototype model of  $L^{1,7,3}$ .

**KEYWORDS:** AdS/CFT correspondence.

---

## Contents

<b>1. Introduction</b>	<b>1</b>
<b>2. The Penrose limit of Sasaki-Einstein <math>Y^{p,q}</math> spaces</b>	<b>3</b>
<b>3. The field theory interpretation</b>	<b>5</b>
<b>4. The <math>L^{p,q,r}</math> spaces case</b>	<b>11</b>
<b>5. The <math>L^{1,7,3}</math> space example</b>	<b>13</b>
<b>6. Conclusions</b>	<b>19</b>

---

## 1. Introduction

During the last two years a series of papers has been published on new infinite families of 5-dimensional Sasaki-Einstein geometries  $Y^{p,q}$  and  $L^{p,q,r}$  [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. The quiver gauge theories (QGT) dual to these backgrounds have been constructed explicitly and analyzed in detail. The results of these papers change the status quo in the gauge/gravity duality, since until recently the only non-trivial superconformal QGT in the context of AdS/CFT was provided by Klebanov and Witten (KW) [12]. The supergravity dual of this model is  $T^{1,1}$ , which appears now to be a special case of the  $Y^{p,q}$  family.

According to the original Maldacena conjecture the chiral operators of the strongly coupled  $\mathcal{N} = 4$   $SU(N)$  gauge theory are in one-to-one correspondence with the modes of type IIB supergravity on  $AdS_5 \times S^5$  [13]. The precise form of the map, however, remains a mystery. One of the main breakthroughs in the study of the correspondence was the idea to consider only states with very large angular momentum along the equator of  $S^5$  [14]. This amounts, effectively, to taking the Penrose limit of  $AdS_5 \times S^5$ . This limit results in a maximally supersymmetric  $pp$ -wave background [15, 16, 17, 18]. Remarkably the string theory in this background is exactly solvable in the light-cone gauge [19, 20]. Combined with the AdS/CFT duality this provides an explicit relation between the dimension and the  $R$ -charge of gauge theory operators dual to the string excitations. These single trace operators with high  $R$ -charge are known as the BMN operators [14].

It appears that for an arbitrary Sasaki-Einstein space  $M_5$ , the Penrose limit around an appropriate null geodesic on  $AdS_5 \times M_5$  results in a maximally supersymmetric  $pp$ -wave background [21]. In particular, this can be done for the conifold background. It implies that, like in the  $\mathcal{N} = 4$  case, apart from a BMN operator corresponding to the string ground state, there are eight additional BMN operators dual to the degenerate first excited state.

In the papers [21, 22, 23] these operators were constructed explicitly in terms of the chiral fields of the KW model. In short, four operators are built by acting with space-time derivatives on the BMN operator dual to the ground state, two additional operators are constructed from the chiral fields in a way similar to the ground state operator, while the last two BMN operators are built from the two  $SU(2)$  currents of the gauge theory. These operators are non-chiral, but still have protected quantum numbers, as can be verified from the supergravity spectroscopy analysis.

Similar analysis was also carried out in [24] for the Klebanov-Strassler [25] and Maldacena-Núñez [26] backgrounds, which are dual to non-conformal  $\mathcal{N} = 1$  supersymmetric gauge theories. Like in the conformal cases the Penrose limits around null geodesics located in the IR region yield exactly solvable string theory models. These represent the non-relativistic motion and low-lying excitations of heavy hadrons with mass proportional to a large global charge. It was further shown in [24] that these hadrons, also termed “annulons”, take the form of heavy non-relativistic strings<sup>1</sup>.

In our paper we take a step further. We take the Penrose limit of the  $Y^{p,q}$  and  $L^{p,q,r}$  backgrounds and analyze the BMN operators of the dual gauge theories. In our analysis we make extensive use of the underlying Kähler quotient structure of the CY cones. It proves to be a very powerful tool for the construction and classification of the chiral gauge invariant operators. We identify the ground state dual operator as well as six chiral BMN operators corresponding to the first excited state in the  $Y^{p,q}$  and the  $L^{p,q,r}$  cases.

Exactly like in the conifold case there are two non-chiral operators dual to the first excited string states. Note, however, that the  $Y^{p,q}$  geometries have only one  $SU(2)$  isometry factor, while the  $L^{p,q,r}$  spaces have no  $SU(2)$  isometry at all. We therefore cannot build the two non-chiral BMN operators entirely from the  $SU(2)$  currents like in the  $T^{1,1}$  case. This problem was first addressed by [6], where the so called “short-cut” non-chiral operator was constructed for the  $Y^{p,q}$  case. Although this non-chiral operator is not a component of any current, it seems to have the right quantum numbers matching the first excited string state. In the  $L^{p,q,r}$  case there are two independent “short-cuts”. In this paper we give a general idea how to build these operators for a general  $L^{p,q,r}$  theory and perform an explicit construction for the  $L^{1,7,3}$  special case.

The outline of the paper is as follows. In Section 2 we show that the Penrose limit of the  $AdS_5 \times Y^{p,q}$  background yields the maximally supersymmetric  $pp$ -wave metric. We also rewrite the light-cone Hamiltonian in terms of the currents and the conformal dimension operator of the dual gauge theory. In Section 3 the BMN construction [21, 22, 23] for the KW model is briefly reviewed. We then rewrite the light-cone Hamiltonian in terms of the derivatives with respect to the Kähler quotient variables of the conifold and reproduce the results of [21, 22, 23]. This method is further used to reconstruct the chiral BMN operators of the  $Y^{p,q}$  theory. We also comment on the short-cut operator of [6]. Section 4 is devoted to the Penrose limit of the  $L^{p,q,r}$  backgrounds. Rather than working with the gauge theory

---

<sup>1</sup>See [27] and [28] for the analogous discussion of the non-supersymmetric deformation of the Klebanov-Strassler background. The “annulons” of the Maldacena-Núñez background and its non-supersymmetric version appear in [28, 29]. For other confining backgrounds see [30, 31]. See also [32] for a general discussion of “annulons” in a confining gauge theory admitting a supergravity dual background.

fields, we work again with the Kähler quotient coordinates, successfully constructing the chiral BMN operators. We end this section with a comment on the “short-cut” operators. In Section 5 we work out the  $L^{1,7,3}$  example providing an explicit construction of the chiral and non-chiral “short-cut” operators. We close in Section 6 with some remarks and suggestions for further research.

## 2. The Penrose limit of Sasaki-Einstein $Y^{p,q}$ spaces

In this section we will construct a maximally supersymmetric  $pp$ -wave background by taking a Penrose limit of the  $\text{AdS}_5 \times Y^{p,q}$  supergravity solution. The global  $\text{AdS}_5$  metric is:

$$\frac{1}{R^2} ds_{\text{AdS}_5}^2 = -dt^2 \cosh^2 \rho + d\rho^2 + \sinh^2 \rho d\Omega_3^2. \quad (2.1)$$

Let us now briefly review the geometry of the Sasaki-Einstein metric on  $Y^{p,q}$ . It is given by [1, 2]:

$$\begin{aligned} \frac{1}{R^2} ds_{Y^{p,q}}^2 = & \frac{1-cy}{6} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{dy^2}{H(y)} + \frac{H(y)}{36} (d\beta + c \cos \theta d\phi)^2 + \\ & + \frac{1}{9} (d\psi' - \cos \theta d\phi + y (d\beta + c \cos \theta d\phi))^2. \end{aligned} \quad (2.2)$$

where

$$H(y) = 2 \frac{a - 3y^2 + 2cy^3}{1 - cy}. \quad (2.3)$$

The conifold case corresponds to  $c = 0$ . Otherwise one can re-scale the coordinate  $y$  to put  $c = 1$ . Written in this way the first line of (2.2) corresponds to the 4d Kähler-Einstein basis parameterized by the coordinates  $\theta, \phi, \beta$  and  $y$ , while the second line is associated with the  $U(1)$ -fibration parameterized by the angle  $\psi'$ . The coordinates  $\theta, \phi$  and  $y$  span the range:

$$0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi \quad \text{and} \quad y_1 \leq y \leq y_2, \quad (2.4)$$

where the constants  $y_{1,2}$  are determined by:

$$y_{1,2} = \frac{1}{4p} \left( 2p \mp 3q - \sqrt{4p^2 - 3q^2} \right). \quad (2.5)$$

To see the periods of  $\beta$  and  $\psi'$  one has to use angles  $\alpha$  and  $\psi$  defined by:

$$\alpha = -\frac{1}{6}(\beta + c\psi') \quad \text{and} \quad \psi = \psi'. \quad (2.6)$$

In these coordinates:

$$0 \leq \alpha < 2\pi\ell, \quad 0 \leq \psi < 2\pi \quad (2.7)$$

with

$$\ell = \frac{q}{3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}}. \quad (2.8)$$

The conifold case corresponds to  $p = 1$ ,  $q = 0$  with  $\ell = 1/3$ . For  $p > 1$ ,  $q = 0$  the metric describes the orbifold of the conifold  $T^{1,1}/\mathcal{Z}_p$  and  $\ell = (3p)^{-1}$ . The  $5d$  compact space  $Y^{p,q}$  has  $SU(2) \times U(1)_F \times U(1)_R$  isometry and its local structure is identical for any  $p, q$ . The only impact of the  $p$  and  $q$  parameters is on the periodicity of the angular coordinate  $\alpha$ . The  $SU(2)$  isometry becomes explicit when the coordinates (2.6) are used. In this case one can conveniently rewrite the  $4d$  Kähler-Einstein metric in terms of the  $SU(2)$  left-invariant Maurer-Cartan forms  $\sigma_{i=1,2,3}$  built from the angles  $\theta, \phi$  and  $\psi$ . The Killing-Reeb vector  $2i\partial_{\psi'}$  is associated with  $R$ -symmetry  $U(1)_R$  [2]. Finally, the invariance with respect to the shift of the  $\alpha$  angle corresponds to the  $U(1)_F$  isometry [4].

The sets of coordinates  $(\theta, \phi)$  and  $(y, \beta)$  describe the base space  $B_4$ , which is topologically the product  $S^2 \times S^2$  [1]. The coordinate  $\alpha$  then corresponds to an  $S^1$  fibration over  $B_4$  and the  $5d$  space is topologically  $S^2 \times S^3$  [1]. To construct a  $pp$ -wave background we will consider a null geodesic lying on the poles of the two spheres. More precisely, we will put  $\theta = 0$  and  $y = y_i$  with  $i = 1, 2$ . This is analogous to the  $T^{1,1}$  example, where the maximally supersymmetric  $pp$ -wave background also emerges in the Penrose limit around a null geodesic located at the poles of the two spheres [21, 22, 23]. In the  $Y^{p,q}$  case there is only one  $SU(2)$  and as a consequence the BMN construction will be different for  $y = y_1$  and  $y = y_2$ . We will use the following coordinate transformation:

$$\begin{aligned} t &= \mu x^+ + \frac{x^-}{\mu R^2} & \rho &= \frac{r}{R} & y &= y_i \left( 1 - 3 \left( \frac{z_1}{R} \right)^2 \right) & \theta &= \left( \frac{6}{1 - cy_i} \right)^{\frac{1}{2}} \frac{z_2}{R} \\ \phi &= -\varphi_2 - \left( \mu x^+ - \frac{x^-}{\mu R^2} \right) & \beta &= \frac{1}{y_i} \varphi_1 + c\varphi_2 + \left( c + \frac{1}{y_i} \right) \left( \mu x^+ - \frac{x^-}{\mu R^2} \right) \\ \psi' &= -\varphi_1 - \varphi_2 + \left( \mu x^+ - \frac{x^-}{\mu R^2} \right). \end{aligned} \quad (2.9)$$

Plugging this into the  $10d$  metric and taking the limit  $R \rightarrow \infty$  we get:

$$ds^2 = -4dx^+ dx^- + dr^2 + r^2 d\Omega_3^2 + dz_1^2 + z_1^2 d\varphi_1^2 + dz_2^2 + z_2^2 d\varphi_2^2 - \mu^2 (r^2 + z_1^2 + z_2^2) dx^{+2}. \quad (2.10)$$

Let us comment on the coordinate transformation (2.9). The transformations of  $t$  and  $\rho$  in (2.9) are standard for backgrounds of the form  $AdS_5 \times M_5$ . Furthermore, the transformations of  $y$  and  $\theta$  are well matched for a null geodesic lying at  $y = y_i$  and  $\theta = 0$ . The unusual  $R^{-2}$  scaling of  $z_1$  can be understood by relating  $y$  to the polar angle  $\zeta$  of the sphere spanned by  $y$  and  $\psi$  [1]:

$$\cos \zeta(y) = \left( \frac{a - 3y^2 + 2cy^3}{a - y^2} \right)^{1/2}. \quad (2.11)$$

It is straightforward to see that expanding around  $\zeta = \pi/2$  (which corresponds to  $y = y_i$ ) we obtain regular  $R^{-1}$  scaling for this coordinate. Let us also comment on the connection to the conifold case. The standard  $T^{1,1}$  coordinates are related to the coordinates of (2.2) by  $\cos \theta_1 = y$ ,  $\theta_2 = \theta$ ,  $\phi_1 = -\beta$  and  $\phi_2 = \phi$ . Substituting  $c = 0$ ,  $a = 3$  and  $y_i = 1$  (corresponding to  $\theta_1 = 0$ ) into (2.9) we recover the transformation of [21, 22, 23].

As was announced in the Introduction (2.10) is the maximally supersymmetric  $pp$ -wave background which preserves all 32 supercharges. Exactly like in the conifold case the supersymmetry is enhanced since the original geometry had only 4 supercharges in 10d. The background is also supported by a non-trivial RR 5-form:

$$F_{(5)} = \mu (r dr \wedge d\Omega_3 + z_1 dz_1 \wedge d\varphi_1 \wedge z_2 dz_2 \wedge d\varphi_2) \wedge dx^+. \quad (2.12)$$

We will close this section by giving a relation between the light-cone world-sheet Hamiltonian for the  $pp$ -wave background (2.10) and the currents associated with the isometries of the original background. We have:

$$\frac{H}{\mu} = -\frac{p_+}{\mu} = i\partial_{x^+} = i\partial_t - i\partial_\phi - \frac{i}{6} \left(1 + \frac{c}{y_i}\right) \partial_\alpha + i \left(\partial_\psi - \frac{c}{6} \partial_\alpha\right). \quad (2.13)$$

First in the global AdS coordinates we have  $i\partial_t = \Delta$ , where  $\Delta$  is the conformal dimension operator. The derivative  $J_3 \equiv -i\partial_\phi$  corresponds to the  $T_3$ -component of the  $SU(2)$  current. Furthermore, we will denote the  $U(1)_F$  charge  $-i\ell\partial_\alpha$  by  $J_\alpha$ . Finally, the  $R$ -symmetry charge  $2i\partial_\psi$  is denoted by  $J_R$ . To summarize, we get:

$$\frac{H}{\mu} = \Delta - J \quad \text{where} \quad J = -J_3 - \frac{1}{6\ell} \left(1 + \frac{1}{y_i}\right) J_\alpha + \frac{1}{2} J_R \quad (2.14)$$

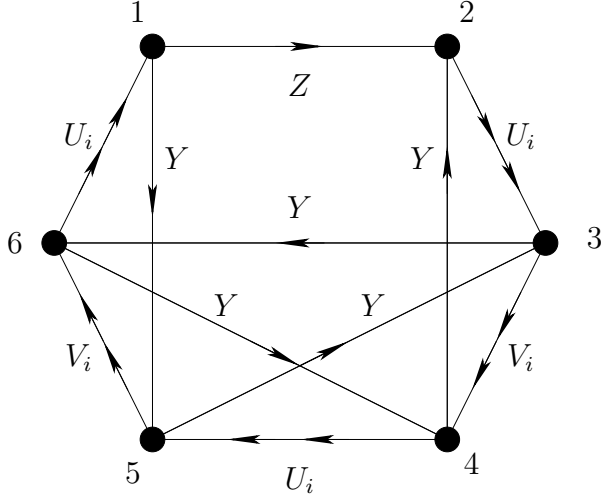
and we from now on we will set  $\mu = 1$ .

### 3. The field theory interpretation

Taking the Penrose limit corresponds to focusing on chiral operators with large  $\Delta$  and  $J$  both scaling like the 't Hooft coupling  $\lambda = g_{YM}^2 N$ , while keeping the light-cone Hamiltonian  $H = \Delta - J$  finite. For a given  $J$  there is a unique light-cone vacuum  $H = 0$ . The corresponding operator in the dual gauge theory has the form  $\text{Tr } \mathcal{O}^J$ , where trace is over the gauge indices. The eight transverse  $H = 1$  excitations of the string are identified by inserting  $\Phi_{i=1,2,3,4}$  and  $D_{a=1,2,3,4} \mathcal{O}$  into the trace [21, 22, 23]. The goal of this section is to find the fields  $\mathcal{O}$  and  $\Phi_{i=1,2,3,4}$  for the case of the  $Y^{p,q}$  field theory dual.

Before proceeding further, let us first briefly review the similar construction [21, 22, 23] for the Klebanov-Witten model [12]. The gauge theory dual to the conifold geometry is coupled to two chiral bi-fundamental multiplets  $(A_+, A_-)$  and  $(B_+, B_-)$ , which transform as a doublet of one of the  $SU(2)$ 's each and are inert under the second  $SU(2)$ . The conifold coordinates are related to these fields in the following way:

$$u = A_+ B_+, \quad v = A_- B_-, \quad x = A_+ B_-, \quad y = A_- B_+ \quad (3.1)$$



**Figure 1:** The quiver diagram of  $Y^{3,2}$ .

and the conifold definition  $uv = xy$  appears as a consistency condition directly following from (3.1). The BMN operator  $\text{Tr}(A_+ B_+)^J$  was identified as the dual to the light-cone Hamiltonian ground state  $H = 0$ . Moving the null geodesic from the north to the south poles of one of the 2-spheres amounts to replacing one of the fields  $A_+$  or  $B_+$  by  $A_-$  or  $B_-$  respectively. The first excited state  $H = 1$  of the world-sheet Hamiltonian is degenerate and there are eight BMN operators corresponding to this state. Six operators are given by:

$$D_{\mu=0,\dots,3} \text{Tr}(A_+ B_+)^J, \quad \text{Tr} A_+ B_- (A_+ B_+)^J \quad \text{and} \quad \text{Tr} A_- B_+ (A_+ B_+)^J, \quad (3.2)$$

while the other two BMN operators are constructed by inserting the lowest components of the two  $SU(2)$  conserved currents into the  $H = 0$  operator  $\text{Tr}(A_+ B_+)^J$ :

$$\text{Tr} A_+ \bar{A}_- (A_+ B_+)^J \quad \text{and} \quad \text{Tr} \bar{B}_+ B_- (A_+ B_+)^J. \quad (3.3)$$

Although these operators are explicitly non-chiral they still have protected dimensions properly matching the  $H = 1$  condition as one can verify by exploring the KK spectrum compactified on  $T^{1,1}$  [33, 21].

The quiver diagram of the gauge theory dual to the  $\text{AdS}_5 \times Y^{p,q}$  supergravity background consists of nodes denoting  $2p$  gauge groups connected by  $4p+2q$  arrows corresponding to various fields in bi-fundamental representations [4]. There are six different types of fields:

- $p$   $SU(2)$  doublets  $U_{\alpha=1,2}$
- $q$   $SU(2)$  doublets  $V_{\alpha=1,2}$
- $p + q$  singlets  $Y$

- $p - q$  singlets  $Z$ .

The quiver diagram for the special case of  $p = 3$  and  $q = 2$  is shown on Fig. 1. The superpotential of the theory is built from various cubic and quartic "blocks" that can be represented symbolically as  $\text{Tr } UVY$  and  $\text{Tr } UZUY$  respectively [4]. In both cases the  $SU(2)$  indices are contracted using the  $\epsilon$ -matrix. The  $F$ -term relations derived from the superpotential produce a set of non-trivial relations among the fields. Using these relations one can construct the chiral ring of the gauge invariant operators [6] (see also [34, 35, 36]). In particular, each of the  $p+q$  superpotential terms (both cubic and quartic) has four gauge invariant operators naturally associated with it, namely operators of the form  $\text{Tr } U_\alpha V_\beta Y$  or  $\text{Tr } U_\alpha Z U_\beta Y$  for  $\alpha, \beta = 1, 2$ . The  $F$ -term conditions imply that all these operators are the same. Moreover, the antisymmetric part of the  $\frac{1}{2} \otimes \frac{1}{2}$  product identically vanishes. Thus we end up with a single spin-1 gauge invariant "short" operator  $\mathcal{S}^{I=-1,0,-1}$  (see Fig. 2). The next chiral primary is obtained by multiplying all of the  $U_\alpha$ ,  $V_\alpha$  and  $Z$  fields in clockwise direction along the quiver. This results in the so-called "long" operator  $\mathcal{L}_+$ . Since the  $F$ -term conditions impose symmetrization over the  $SU(2)$  indices, the only non-trivial component of this operator transforms in the  $\frac{1}{2}(p+q)$  representation of  $SU(2)$ . Finally, there is an additional "long" operator  $\mathcal{L}_-$  built from the  $U_\alpha$  and  $Y$  fields. It transforms in the  $\frac{1}{2}(p-q)$  representation (see Fig. 3). Remarkably, the operators  $\mathcal{L}_+$  and  $\mathcal{L}_-$  have winding numbers +1 and 0 with respect to the quiver diagram.

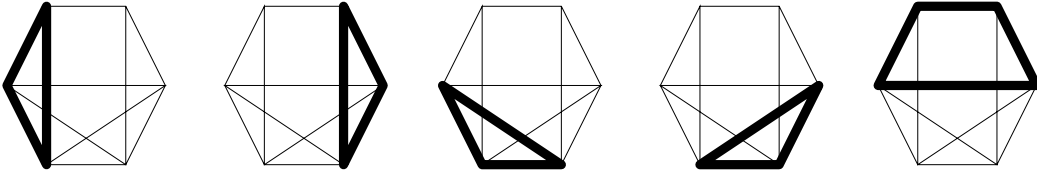
The charges of the operators are given by [6]:

	spin $J$	$J_R$	$J_\alpha$
$\mathcal{S}^I$	1	2	0
$\mathcal{L}_+$	$\frac{p+q}{2}$	$p+q - \frac{1}{3\ell}$	1
$\mathcal{L}_-$	$\frac{p-q}{2}$	$p-q + \frac{1}{3\ell}$	-1

(3.4)

Substituting the charges of the lowest  $SU(2)$  component ( $J_3 = -\frac{1}{2}(p+q)$ ) of the  $\mathcal{L}_+$  operator into the Hamiltonian (2.14) we easily find that  $H = 0$  if  $y = y_1$ . Therefore the string vacuum state in this case corresponds to the operator  $\text{Tr } \mathcal{L}_+^J$ . Analogously for  $y = y_2$  the relevant operator is  $\text{Tr } \mathcal{L}_-^J$ . In verifying this statement it is useful to recall that for the chiral primaries  $\Delta = \frac{3}{2}|J_R|$  and:

$$p+q - \frac{1}{3\ell} = -\frac{1}{3\ell} \frac{1}{y_1} \quad \text{and} \quad p-q + \frac{1}{3\ell} = \frac{1}{3\ell} \frac{1}{y_2}. \quad (3.5)$$



**Figure 2:** All  $UVY$  and  $UZUY$  "short" operators of the  $Y^{3,2}$  quiver theory. The  $F$ -term conditions imply that all these operators are equivalent [6].



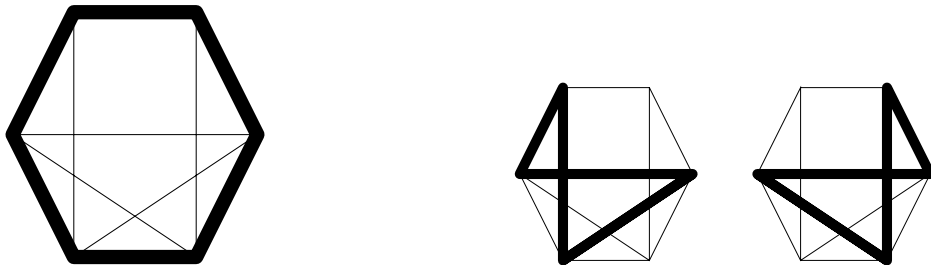
Next let us consider the insertions corresponding to the eight  $H = 1$  states. As we have mentioned above four insertion operators are given by space derivatives  $D_\mu \mathcal{L}_+$  and  $D_\mu \mathcal{L}_-$  respectively. Therefore we have to identify four additional operators:

1. We can obtain  $H = 1$  by inserting the “short” operator  $\mathcal{S}^{I=-1}$  as it follows immediately from the table (note that for  $\mathcal{S}^{I=-1}$  we have  $J_3 = -1$ ).
2. We took for the  $H = 0$  BMN operators the lowest components of the  $\frac{1}{2}(p+q)$  and  $\frac{1}{2}(p-q)$   $SU(2)$  representations related to the  $\mathcal{L}_+$  and  $\mathcal{L}_-$  operators respectively. It is natural therefore to consider a “spin flip” operator: we can change the spin of one of the doublets along the “long” operator then symmetrizing over all possible “flips”. Since for the modified operator  $\delta J_3 = 1$  with all other charges unchanged we find that it matches perfectly the  $H = 1$  condition. This is analogous to the  $A_+ \rightarrow A_-$  and  $B_+ \rightarrow B_-$  flips in the conifold case.
3. Consider the lowest component of the conserved  $SU(2)$  current:

$$\mathcal{K}_{SU(2)}^I = \sum_i \sigma_{\alpha\beta}^I \text{Tr} \left( U_\alpha^{i,i+1} \bar{U}_\beta^{i+1,i} + V_\alpha^{i,i+1} \bar{V}_\beta^{i+1,i} \right), \quad (3.6)$$

where  $\sigma_{\alpha\beta}^{I=1,2,3}$  are the Pauli matrices. This operator has protected dimension  $\Delta = 2$  and vanishing  $J_R$  and  $J_\alpha$ . On the other hand  $J_3 = -1, 0, 1$  and taking the lowest component ( $J_3 = -1$ ) as an insertion we find the required  $H = 1$  result. Again, as in the  $T^{1,1}$  case, there is no apparent field theory argument protecting the naive counting and we have to analyze the supergravity spectrum with the given quantum numbers in order to verify the prediction. Unfortunately for an arbitrary  $Y^{p,q}$  background it is quite difficult to carry out these calculations (for related discussions of the issue see [37] and [38]).

4. The last operator can be produced using the “short-cut” operator [6]. One starts from the lowest component of “long” operator  $\mathcal{L}_+$ , replaces a fragment  $V_2^{i,i+1} U_2^{i+1,i+2}$ ,  $U_2^{i,i+1} V_2^{i+1,i+2}$  or  $U_2^{i,i+1} Z U_2^{i+1,i+2}$  by the nearby antichiral  $\bar{Y}_{i,i+2}$  field and finally symmetrizes by the “replacement” all over the quiver. For the new non-chiral operator we have  $\delta J_\alpha = 0$ ,  $\delta J_R = -2$  and  $\delta J_3 = -1$  (recall that we have replaced a symmetrized



**Figure 3:** The  $\mathcal{L}_+$  “long” (left) and the  $\mathcal{L}_-$  “long” (right) operator of  $Y^{3,2}$ .

product of two  $SU(2)$  doublets by a singlet). Furthermore, we have  $\delta\Delta = -1$ . This might be expected from the fact that the “short-cut” can be thought of as a combination of an insertion of the  $U(1)_\alpha$  current  $\mathcal{K}_\alpha = \sum Y^{i,i+2} \bar{Y}^{i+2,i} + \dots$  and a removal of  $\mathcal{S}^{I=-1}$ . It is easy to “verify” that again  $H = 1$ . Alternatively, for the case with the null geodesic at  $y = y_2$  we can produce the  $H = 1$  operator by replacing one of the fragments  $U_2^{i,i+1} Y^{i+1,i-1}$  of the “long” operator  $\mathcal{L}^-$  (see Fig. 3) by the corresponding anti-chiral field  $\bar{V}^{i+2,i}$ . Again, the complexity of the  $Y^{p,q}$  background prevents us from verifying this result by supergravity spectrum analysis and we refer the reader to the related papers [37] and [38].

To summarize, the structure of the BMN operators of the  $Y^{p,q}$  theories is quite similar to the analogous construction in the Klebanov-Witten  $T^{1,1}$  model. Four out of eight operators corresponding to the  $H = 1$  excited state are obtained by applying space-time derivatives on the ground state operator. Two additional operators are produced by the spin “flip” and the  $SU(2)$  current insertion exactly like in the conifold example. The last two operators (the “short” operator insertion and the “short-cut”) differ, however, from the BMN construction in the KW model. This of course is a remnant of the fact that there is only one  $SU(2)$  factor in the symmetry group of the  $Y^{p,q}$  model.

Notice that from four “non-derivative” operators two are chiral and the other two are non-chiral precisely like in the conifold case. In the rest of the section we will show that there is a straightforward way to identify these chiral BMN operators using the fact that a Calabi-Yau cone over  $Y^{p,q}$  is actually a Kähler quotient  $\mathbb{C}^4 // U(1)$ , namely a gauged linear  $\sigma$ -model (GLSM) with  $U(1)$  charges  $(p, p, -p+q, -p-q)$  [2]. As one of the checks, exploring the chiral ring relations (as was briefly outlined above) one arrives at the conclusion, that all the gauge-invariant chiral operators of the theory are in one-to-one correspondence with  $U(1)$ -invariant polynomials of the GLSM. These polynomials are of the form:

$$P = w_1^{n_1} w_2^{n_2} w_3^{n_3} w_4^{n_4}, \quad (3.7)$$

where  $w_i$ ’s are the  $\mathbb{C}^4$  coordinates and the non-negative integers  $n_i$ ’s satisfy the  $U(1)$ -invariance condition:

$$p(n_1 + n_2) - (p - q)n_3 - (p + q)n_4 = 0. \quad (3.8)$$

In particular, in the conifold case the coordinates  $w_i$  may be identified directly with the four fields  $A_\pm$  and  $B_\pm$ . This just reflects the fact that  $F$ -term conditions derived from the Klebanov-Witten superpotential don’t impose non-trivial relations between the fields. For an arbitrary  $Y^{p,q}$  there is certainly no direct link between the fields  $U_\alpha, V_\alpha, Y, Z$  and the  $w_i$  coordinates of the corresponding GLSM, and we can only map gauge invariant products of the fields to the polynomials of the form (3.7). There are three types of independent polynomials for any  $p$  and  $q$ :

1.  $a_k = w_1^k w_2^{p-q-k} w_3^p$  with  $k = 0, \dots, p - q$ . This corresponds to the  $(p - q + 1)$  components of the “long” operator  $\mathcal{L}_-$  discussed above.

2.  $b_1 = w_1^2 w_3 w_4$ ,  $b_2 = w_1 w_2 w_3 w_4$  and  $b_3 = w_2^2 w_3 w_4$ . These are the three components of the “short” operator  $\mathcal{S}^{I=-1,0,1}$ .
3.  $c_k = w_1^k w_2^{p+q-k} w_4^p$  with  $k = 0, \dots, p+q$ . This corresponds to the “long” operator  $\mathcal{L}_+$ .

Next let us denote  $\theta_i = \text{Arg}(w_i)$ . Note that  $\partial_{\theta_i} = -in_i$  while acting on the polynomials of the form (3.7). Moreover, the derivatives  $\partial_{\theta_i}$  can be expressed in terms of the derivatives with respect to the angular coordinates appearing in the metric of  $Y^{p,q}$  (see [2] for the detailed explanation):

$$\begin{aligned}
\partial_{\theta_1} &= \partial_\phi + \partial_\psi \\
\partial_{\theta_2} &= -\partial_\phi + \partial_\psi \\
\partial_{\theta_3} &= \partial_\psi - \frac{\ell}{2}(p+q)\partial_\alpha \\
\partial_{\theta_4} &= \partial_\psi + \frac{\ell}{2}(p-q)\partial_\alpha.
\end{aligned} \tag{3.9}$$

We can use these identities to express the derivatives  $\partial_\phi$ ,  $\partial_\psi$  and  $\partial_\alpha$  in terms of the derivatives  $\partial_{\theta_i}$ ’s. Substituting further these relations into the expressions for  $J_R$  and  $J$  (see (2.14)) we can re-write these currents solely in terms of the numbers  $n_i = -i\partial_{\theta_i}$ . Finally, since for the chiral primary operators  $\Delta = \frac{3}{2}|J_R|$  we obtain the following simple identity for  $H = \Delta - J$  (we will put  $\mu = 1$ ):

$$H_{\text{C.P.}} = n_1 + n_3 \quad \text{if } y = y_1 \quad \text{and} \quad H_{\text{C.P.}} = n_1 + n_4 \quad \text{if } y = y_2. \tag{3.10}$$

Here the subscript “C.P.” reminds again that the relation is valid only for chiral primary operators and we used (3.5) in the calculations.

We are now in a position to verify our results for the BMN operators dual to the  $H = 0$  and  $H = 1$  string states. For simplicity let us consider the  $y = y_1$  case. Since all  $n_i$  in (3.7) are non-negative,  $H_{\text{C.P.}} = 0$  iff  $n_1, n_3 = 0$ . The only polynomial of this form is  $c_0^N$  for arbitrary  $N$ . Since  $c_0 = w_2^{p+q} w_4^p$  is associated with the lowest component of the “long” operator  $\mathcal{L}_+$ , we successfully reproduce our result for the BMN operator dual to the ground state. Furthermore, there are two options for the  $H_{\text{C.P.}} = 1$  state. Namely, for  $n_1 = 1$ ,  $n_3 = 0$  the corresponding polynomial is  $c_1 = w_1 w_2^{p+q-1} w_4^p$  and for  $n_1 = 0$ ,  $n_3 = 1$  the polynomial is  $b_3 = w_2^2 w_3 w_4$ . Since the former corresponds to the spin flip and the latter to the lowest component of the “short” operator  $\mathcal{S}^{I=-1}$  we recover the above-mentioned result for the two chiral  $H = 1$  states. Finally, let us address the “short-cut”  $H = 1$  non-chiral operator. As we have just discussed this operator is obtained by the current  $\mathcal{K}_\alpha$  insertion into the  $\text{Tr } \mathcal{L}_+^J$  string followed by the  $\mathcal{S}^{I=-1}$  operator removal. From the current insertion we get  $\delta H = 2$ , so we need  $\delta H = 1$  for the chiral operator  $\mathcal{S}^{I=-1}$ . From the discussion above this is clearly the case, since  $\mathcal{S}^{I=-1}$  corresponds to the  $b_3 = w_2^2 w_3 w_4$  polynomial with  $n_1 + n_3 = 1$ . The reader may wonder if it is possible to construct another “short-cut”  $H = 1$  operator starting from the same current, but removing another chiral operator from the

string (for instance,  $S^{I=+1}$ ). An easy check reveals, however, that the  $S^{I=-1}$  “short-cut” is the only possibility.

#### 4. The $L^{p,q,r}$ spaces case

In this section we will apply the method proposed above to the  $L^{p,q,r}$  case. This is a larger family of backgrounds with only  $U(1)^3$  isometry group, which include the  $Y^{p,q}$  sub-family as a special case. For general  $p, q$  and  $r$  the gauge theory field content is extremely complicated and we will not try to present it here (see [7, 10, 8]). Instead, we will use the underlying Kähler quotient structure of the space exactly as we did for  $Y^{p,q}$  in the previous section. First we will show that the Penrose limit again provides the  $pp$ -wave metric (2.10). Then we will re-write the light-cone Hamiltonian in terms of the derivatives (3.9) and will use this presentation to identify the BMN operator dual to the ground state and two chiral “non-derivative” BMN operators corresponding to the first excited state.

We will start with a very brief review of the  $L^{p,q,r}$  geometry. The relevant 5d Sasaki-Einstein metric is [5, 11]:

$$\begin{aligned} \frac{1}{R^2} ds_{L^{p,q,r}}^2 = & (d\tau + \sigma)^2 + \frac{\rho^2 dx^2}{4\Delta_x} + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \frac{\Delta_x}{\rho^2} \left( \frac{\sin^2 \theta}{\alpha} d\phi + \frac{\cos^2 \theta}{\beta} d\psi \right)^2 + \\ & + \frac{\Delta_\theta \sin^2 \theta \cos^2 \theta}{\rho^2} \left( \frac{\alpha - x}{\alpha} d\phi - \frac{\beta - x}{\beta} d\psi \right)^2, \end{aligned} \quad (4.1)$$

where

$$\begin{aligned} \sigma = & \frac{(\alpha - x) \sin^2 \theta}{\alpha} d\phi + \frac{(\beta - x) \cos^2 \theta}{\beta} d\psi, \\ \Delta_x = & x(\alpha - x)(\beta - x) - 1, \quad \Delta_\theta = \alpha \cos^2 \theta + \beta \sin^2 \theta \quad \text{and} \quad \rho^2 = \Delta_\theta - x. \end{aligned} \quad (4.2)$$

The constants  $\alpha$  and  $\beta$  as well as the period of the angular coordinate  $\tau$  are very complicated functions of the three co-prime integer parameters  $p, q$  and  $r$  [9, 11] satisfying  $p < r < q$ . In particular, for  $p + q = 2r$  one has  $\alpha = \beta$  and the geometry reduces to the  $Y^{p',q'}$  case with  $p' = \frac{1}{2}(p + q)$  and  $q' = \frac{1}{2}(q - p)$ . The  $x$  coordinate ranges between  $x_1$  and  $x_2$ , the lowest two roots of the equation  $\Delta_x = 0$ . Moreover,  $0 \leq \theta \leq \frac{\pi}{2}$ .

The 4-dimensional base of the  $L^{p,q,r}$  space is topologically the product  $S^2 \times S^2$  exactly like in the  $Y^{p,q}$  case. The coordinates  $x$  and  $\theta$  are the azimuthal coordinates on the two 2-spheres. We will again assume that the null geodesic is located at the poles of the spheres, namely  $x = x_1$  or  $x_2$  and  $\theta = 0$  or  $\frac{\pi}{2}$  along the geodesic. Since there is no  $SU(2)$  isometry like in the  $Y^{p,q}$  geometry, taking the Penrose limit might yield four different interpretations on the field theory side depending on the four possible locations of the geodesic. For the sake of simplicity in what follows we will only consider the  $\theta = 0$  option.

The coordinate transformation we will use is:

$$\begin{aligned}
x &= x_i + \frac{\Delta'_i}{\alpha - x_i} \frac{z_1^2}{R^2} & \theta &= \left( \frac{\alpha}{\alpha - x_i} \right)^{\frac{1}{2}} \frac{z_2}{R} \\
\phi &= a_i \varphi_1 + \varphi_2 + (1 + a_i) \left( \mu x^+ - \frac{x^-}{\mu R^2} \right) \\
\psi &= b_i \varphi_1 + b_i \left( \mu x^+ - \frac{x^-}{\mu R^2} \right) \\
\tau &= c_i \varphi_1 + (1 + c_i) \left( \mu x^+ - \frac{x^-}{\mu R^2} \right),
\end{aligned} \tag{4.3}$$

where  $i = 1$  or  $2$  and the transformation of the  $AdS_5$  coordinates is the same as in the  $Y^{p,q}$  case. The constants  $a_i$ ,  $b_i$  and  $c_i$  are given by:

$$a_i = \frac{\alpha(\beta - x_i)}{\Delta'_i}, \quad b_i = \frac{\beta(\alpha - x_i)}{\Delta'_i}, \quad c_i = -\frac{(\beta - x_i)(\alpha - x_i)}{\Delta'_i} \quad \text{with} \quad \Delta'_i \equiv \left. \frac{\partial \Delta_x}{\partial x} \right|_{x=x_i}. \tag{4.4}$$

From 4.3 we find that  $\partial_{\varphi_1} = l_i$ , where  $l_i = a_i \partial_\phi + b_i \partial_\psi + c_i \partial_\tau$  is the Killing vector whose length vanishes at  $x = x_i$  [5]. Similarly  $\partial_{\varphi_1} = \partial_\phi$  is the Killing vector whose norm equal to zero at  $\theta = 0$ . It means that if all triples among the integers  $(p, q, r, s)$  are co-prime, and as a consequence there will be no conical singularities, than the periods of both  $\varphi_1$  and  $\varphi_2$  will be  $2\pi$ .

Substituting (4.3) into the 10d metric and taking the  $R \rightarrow \infty$  limit yields the maximally supersymmetric  $pp$ -wave background (2.10) and the light-cone world-sheet Hamiltonian is given by:

$$\frac{H}{\mu} = i \partial_{x^+} = \Delta - J, \tag{4.5}$$

where  $\Delta = i \partial_t$  as usual and

$$J = -i b_i \partial_\psi - i(1 + a_i) \partial_\phi - i(1 + c_i) \partial_\tau. \tag{4.6}$$

Now we will use the method described in the previous section in order to identify the chiral BMN operators dual to the ground and the first excited states of the light-cone Hamiltonian (4.5). A Calabi-Yau cone over the  $L^{p,q,r}$  base is a Kähler quotient with charges  $(p, q, -r, r - p - q)$  and the set of four Killing vectors analogous to the set (3.9) is given by [9]:

$$\begin{aligned}
\partial_{\theta_i} &= -(c_i \partial_\tau + a_i \partial_\phi + b_i \partial_\psi) & \text{for } i &= 1, 2 \\
\partial_{\theta_3} &= -\partial_\phi \\
\partial_{\theta_4} &= -\partial_\psi.
\end{aligned} \tag{4.7}$$

Furthermore, the  $R$ -charge current is:

$$J_R = -\frac{2}{3}i\partial_\tau. \quad (4.8)$$

Plugging these identities into (4.5) and recalling again that for chiral primary operators  $\Delta = \frac{3}{2}|J_R|$  we arrive at the following result for  $\mu = 1$ :

$$H_{\text{C.P.}} = n_1 + n_3 \quad \text{if } x = x_1 \quad \text{and} \quad H_{\text{C.P.}} = n_2 + n_3 \quad \text{if } x = x_2. \quad (4.9)$$

Notice that for a null geodesic lying at  $\theta = \frac{\pi}{2}$  we would get the same result with  $n_3$  replaced by  $n_4$ .

It is now a straightforward exercise to find the polynomials of  $w_i$ 's, which will correspond to  $H = 0$  and  $H = 1$ . Let us focus on the  $x = x_1$  case. As in the  $Y^{p,q}$  case the condition  $H_{\text{C.P.}} = 0$  implies that  $n_1 = n_3 = 0$  and hence the relevant polynomial is  $w_2^s w_4^q$  with  $s = p + q - r$ . Finally, for  $H_{\text{C.P.}} = 1$  we have polynomials corresponding to  $(n_1, n_3) = (1, 0)$  and  $(n_1, n_3) = (0, 1)$ . The first polynomial is  $w_1 w_2^a w_4^b$  with  $p + qa - sb = 0$  and the second is  $w_3 w_2^c w_4^d$  with  $r + qc - sd = 0$ . Notice that in both cases the solutions of the Euclidean equations exist, since the integers  $q$  and  $s$  are co-prime. As we explained above, for not co-prime  $q$  and  $s$  the period of the angular coordinates  $\varphi_1$  and  $\varphi_2$  will be different from  $2\pi$  changing the string spectrum in the  $pp$ -wave background (2.10)<sup>2</sup>.

Next let us address the remaining two non-chiral operators corresponding to the  $H = 1$  state. Unlike in the  $Y^{p,q}$  case here we do not have  $SU(2)$  symmetry and we therefore cannot use the related current as an insertion in order to construct the relevant  $H = 1$  operator. On the other hand, we have two independent  $U(1)$  currents and so we might attempt to build two appropriate “short-cut” non-chiral operators for each one of the currents. Exactly like in the  $Y^{p,q}$  case we have  $\delta H = 2$  for these currents, since they are invariant under the  $U(1)$  isometries of the theory, while  $\Delta = 2$  by the field theory arguments. It implies again that we are looking for two chiral operators with  $\delta H_{\text{C.P.}} = 1$  or  $n_1 + n_3 = 1$ . These are actually precisely the  $H = 1$  operators corresponding to the polynomials  $(n_1, n_3) = (1, 0)$  and  $(n_1, n_3) = (0, 1)$  that we have discussed in the previous paragraph. This statement, however, still needs to be verified by an explicit construction in terms of the field theory operators similarly to what we did in the  $Y^{p,q}$  case.

## 5. The $L^{1,7,3}$ space example

Unfortunately, we were not able to construct explicitly the BMN operators dual to the ground and the first excited states for arbitrary parameters  $(p, q, r, s)$ . Instead we will thoroughly analyze the  $L^{1,7,3}$  example ( $p = 1, q = 7, r = 3$  and  $s = 5$ ). To this end we have to identify the polynomials of the GLSM with the charges  $(1, 7, -3, -5)$  in terms of the gauge invariant field theory operators.

---

<sup>2</sup>In this case the calculation of the spectrum will be similar to the Penrose limit of the orbifold of  $AdS_5 \times S^5$  (see [39, 40] and also [41].)

As in the previous sections we will denote the Kähler quotient coordinates by  $w_1$ ,  $w_2$ ,  $w_3$  and  $w_4$  with the charges 1, 7,  $-3$  and  $-5$  respectively. A simple straightforward calculation shows that for these charges one has 12 independent polynomials<sup>3</sup> :

$$\begin{aligned} a &= w_1^3 w_3, & b &= w_2^5 w_4, & c &= w_1^5 w_4, & d &= w_2^3 w_3, & e &= w_1 w_2 w_3 w_4 \\ f_1 &= w_1 w_2^2 w_3^5, & f_2 &= w_1 w_2^2 w_4^3, & f_3 &= w_1^2 w_2 w_3^3, & f_4 &= w_1^3 w_2 w_4^2, \\ f_5 &= w_2^4 w_3 w_4^5, & f_6 &= w_2^2 w_3^3 w_4, & f_7 &= w_2^3 w_3^2 w_4^3. \end{aligned} \quad (5.1)$$

Since the complex space described by the variables is only 3-dimensional, these variables are subject to many redundant relations between them. Here we will list only a few of them:

$$a^{35} b^3 = c^{21} d^5, \quad (5.2)$$

$$e^8 = abcd, \quad (5.3)$$

$$f_1^3 = ad^2, \quad f_2^5 = cb^2, \quad f_3^3 = a^2 d, \quad f_4^5 = c^3 b, \quad f_5^7 = b^5 d, \quad f_6^7 = bd^3, \quad f_7^7 = b^3 d^2. \quad (5.4)$$

The relations (5.2) and (5.3) can be easily generalized to arbitrary parameters  $p$ ,  $q$ ,  $r$  and  $s = p + q - r$ . Indeed, defining:

$$a = w_1^r w_3^p, \quad b = w_2^s w_4^q, \quad c = w_1^s w_4^p, \quad d = w_2^r w_3^q, \quad e = w_1 w_2 w_3 w_4 \quad (5.5)$$

we get:

$$a^{qs} b^{pr} = c^{qr} d^{ps} \quad \text{and} \quad e^{p+q} = abcd. \quad (5.6)$$

Let us next briefly review the field theory content and the superpotential of the gauge theory dual of an arbitrary  $L^{p,q,r}$  background. There are six types of fields that we will denote<sup>4</sup> by  $U_1$ ,  $U_2$ ,  $V_1$ ,  $V_2$ ,  $Y$  and  $Z$  following the  $Y^{p,q}$  conventions. The multiplicities of these fields are given by:

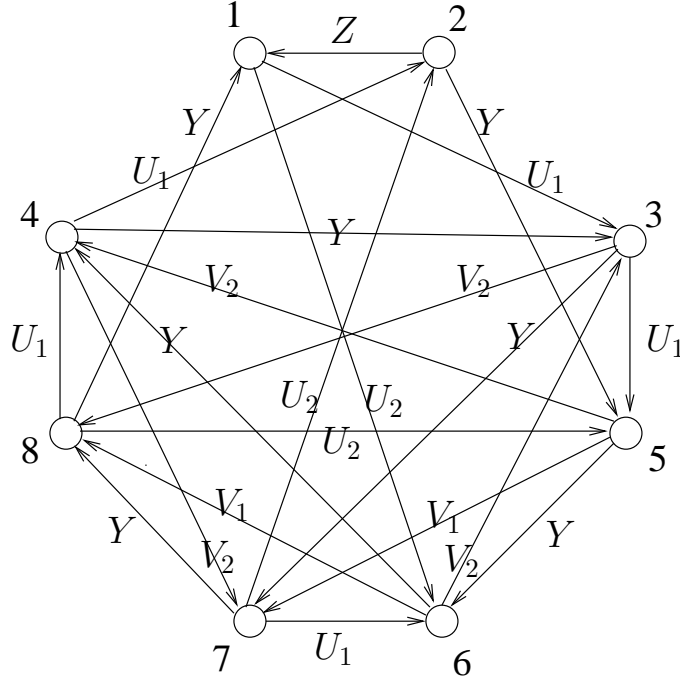
$$\begin{aligned} \text{mult}[U_1] &= s, & \text{mult}[U_2] &= r, & \text{mult}[V_1] &= r - p, & \text{mult}[V_2] &= q - r, \\ \text{mult}[Y] &= q, & \text{mult}[Z] &= p. \end{aligned} \quad (5.7)$$

In particular, for  $r = s$  the multiplicities of  $U_1$  and  $U_2$  become equal reproducing correctly the number of the  $SU(2)$  doublets  $(U_1, U_2)$  in the  $Y^{p',q'}$  theory with  $p' = \frac{1}{2}(p + q) = r$ . Similar checks can be easily performed for the fields  $V_1$  and  $V_2$ . The quiver diagram of  $L^{1,7,3}$  is depicted in Fig. 4. Furthermore, exactly like in the  $Y^{p,q}$  case there are three types of polynomials constructed from these fields that may appear in the superpotential:

---

<sup>3</sup>By independence of the polynomials we mean that none of them can be written in terms of other polynomials from the list, namely there is no relation of the form  $p = p_1^{n_1} p_2^{n_2} \dots$

<sup>4</sup>In this paper we will follow mostly the notations of [8] exchanging only the fields  $V_1$  and  $V_2$  in order to make explicit the similarity to the  $Y^{p,q}$  case.



**Figure 4:** The quiver diagram of  $L^{1,7,3}$  ( $p = 1, q = 7, r = 3$  and  $s = 5$ ).

$$W_0 = \text{Tr } Y U_1 Z U_2, \quad W_1 = \text{Tr } Y U_1 V_2 \quad \text{and} \quad W_2 = \text{Tr } Y U_2 V_1. \quad (5.8)$$

The number each term appears in the superpotential are  $2p$ ,  $2(q - r)$  and  $2(r - p)$  respectively. The explicit form of the superpotential is quite complicated, but it can be figured out in a straightforward manner using the corresponding dimer tiling (see [8]). For the given  $L^{1,7,3}$  example, however, the superpotential blocks (5.8) can be read directly from the quiver diagram on Fig. 4.

Differentiating the superpotential with respect to the fields we will obtain a set of  $F$ -term conditions. Unlike in the conifold example these conditions are non-trivial and impose relations between various operators constructed from the fields. The task of constructing all possible gauge invariant field polynomials (operators) is very cumbersome already for the  $L^{1,7,3}$  case, and here we will report only the final results.

First, there are four polynomials analogous to the “long” operators  $\mathcal{L}_{\pm}$  which appeared in the  $Y^{p,q}$  diagrams (see Fig. 3). They can be written in a schematic way as:

$$\begin{aligned} \mathcal{L}_+^{\uparrow} &= \text{Tr } Z^p (U_1)^s (V_1)^{q-s}, & \mathcal{L}_+^{\downarrow} &= \text{Tr } Z^p (U_2)^r (V_2)^{q-r}, \\ \mathcal{L}_-^{\uparrow} &= \text{Tr } (U_1)^p Y^r, & \mathcal{L}_-^{\downarrow} &= \text{Tr } (U_2)^p Y^s. \end{aligned} \quad (5.9)$$

Here the arrows indicate that for  $r = s$  the operators  $\mathcal{L}_{\pm}^{\uparrow}$  reduce to the highest  $SU(2)$  components of the “long” operators  $\mathcal{L}_{\pm}$  of the related  $Y^{p',q'}$  theory. Similarly, the operators



$\mathcal{L}_\pm^\downarrow$  become the lowest components of  $\mathcal{L}_\pm$ . Using the diagram in Fig. 4 it is quite easy to construct the  $\mathcal{L}_+^\uparrow$  and  $\mathcal{L}_+^\downarrow$  operators explicitly for the  $L^{1,7,3}$  case:

$$\mathcal{L}_+^\uparrow = \text{Tr } Z_{21} U_{16}^2 V_{63}^2 V_{38}^2 U_{85}^2 V_{54}^2 V_{47}^2 U_{72}^2, \quad \mathcal{L}_+^\downarrow = \text{Tr } Z_{21} U_{13}^1 U_{35}^1 V_{57}^1 U_{76}^1 V_{68}^1 U_{84}^1 U_{42}^1. \quad (5.10)$$

Like in the  $Y^{p,q}$  case these two operators have single representations in terms of the fields  $U_i, V_i, Y$  and  $Z$ . The operators  $\mathcal{L}_-^\uparrow$  and  $\mathcal{L}_-^\downarrow$ , however, have many possible representations, again similar to the  $Y^{p,q}$  example. For instance, the operator  $\mathcal{L}_-^\uparrow$  may be written in five equivalent ways as can be shown by analyzing the set of  $F$ -term conditions:

$$\begin{aligned} \mathcal{L}_-^\uparrow = & \text{Tr } U_{84}^1 Y_{43} Y_{37} Y_{78}, \quad \text{Tr } U_{13}^1 Y_{37} Y_{78} Y_{81}, \quad \text{Tr } U_{42}^1 Y_{25} Y_{56} Y_{64}, \\ & \text{Tr } U_{76}^1 Y_{64} Y_{43} Y_{37}, \quad \text{Tr } U_{35}^1 Y_{56} Y_{64} Y_{43}. \end{aligned} \quad (5.11)$$

Using the  $F$ -term relations it is quite easy to show that these four operators are equivalent to each other. For example, in order to prove the equivalence of the first two operators in (5.11) it is enough to replace  $U_{84}^1 Y_{43}$  by  $Y_{81} U_{13}^1$  using the  $F$ -term condition for the field  $V_{38}^2$ . The operator  $\mathcal{L}_-^\downarrow$  also can be presented in various ways:

$$\mathcal{L}_-^\downarrow = \text{Tr } U_{85}^2 Y_{56} Y_{64} Y_{43} Y_{37} Y_{78}, \quad \text{Tr } U_{16}^2 Y_{64} Y_{43} Y_{37} Y_{78} Y_{81}, \quad \text{Tr } U_{72}^2 Y_{25} Y_{56} Y_{64} Y_{43} Y_{37}. \quad (5.12)$$

The operators  $\mathcal{L}_\pm^\uparrow$  and  $\mathcal{L}_\pm^\downarrow$  were called extremal BPS mesons in [7]. Indeed, it can be verified that these operators have maximal  $U(1)$  charges (in modulus) for given  $R$ -charge. Furthermore, it was argued in [7] that these four mesons correspond to the BPS geodesics, which lie at the vertices of the coordinate rectangular, namely at  $x = x_1$  or  $x_2$  and  $\theta = 0$  or  $\frac{\pi}{2}$ . In terms of the Kähler quotient coordinates the vertices are given by  $w_i = w_j = 0$ , where  $i = 1$  or  $2$  and  $j = 3$  or  $4$ . Therefore these vertices are well described by the polynomials  $a, b, c$  and  $d$  from (5.1). For example, for  $w_2 = w_4 = 0$  the only non-vanishing polynomial in (5.1) is  $a$ . Thus we find that it is natural to relate the extremal BPS mesons  $\mathcal{L}_\pm^\uparrow$  and  $\mathcal{L}_\pm^\downarrow$  to the variables  $a, b, c$  and  $d$ . To be more precise the identification is as follows:

$$\mathcal{L}_+^\uparrow \longleftrightarrow d, \quad \mathcal{L}_+^\downarrow \longleftrightarrow b, \quad \mathcal{L}_-^\uparrow \longleftrightarrow a, \quad \mathcal{L}_-^\downarrow \longleftrightarrow c. \quad (5.13)$$

To prove this statement one can verify, for instance, that  $R$ -charges of the corresponding operators and polynomials in (5.13) coincide with each other. Alternatively we can check (5.13) by substituting the operators  $\mathcal{L}_\pm^{\uparrow\downarrow}$  instead of  $a, b, c$  and  $d$  into the relation (5.2) and proving it using the  $F$ -term conditions. We will not perform this tedious calculation here. Instead we will confirm (5.2) by examining the quantum numbers of the operators. Let us assign the following quantum numbers to the gauge theory fields:

	$U_1$	$U_2$	$V_1$	$V_2$	$Z$	$Y$
$Q_1$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
$Q_2$	1	1	1	1	0	0
$Q_3$	0	0	0	0	0	1

(5.14)

Clearly, the superpotential is invariant under these  $U(1)$  symmetries, since all of the superpotential blocks in (5.8) have the same charges  $(0, 2, 1)$  with respect to (5.14). These  $U(1)$ 's are actually linear combinations of the  $U(1)_R$ ,  $U(1)_B$  and the other two  $U(1)$  global symmetries of [7, 8, 10], but for what follows we will not need any relation between the symmetries of [7, 8, 10] and the charges of the table (5.14). Substituting these charges into (5.9) we will get the charges of the “long” operators  $\mathcal{L}_\pm^\uparrow$  and  $\mathcal{L}_\pm^\downarrow$ . It is now a simple exercise to confirm that with the identification (5.13) the left and the right hand sides of (5.2) (or (5.6) for arbitrary  $p, q$  and  $r$ ) have the same charges. This provides a very non-trivial check of (5.13).

We can use the same method in order to find operators corresponding to the variables  $e$  and  $f_{i=1\dots 7}$ . Indeed, we see from (5.3) that the  $U(1)$  numbers (5.14) of the operators corresponding to  $e$  are  $(0, 2, 1)$ . These operators, therefore, are just the superpotential blocks (5.8). There are  $2q$  blocks in general and their equivalence almost trivially follows from the  $F$ -term conditions. Remarkably, there are precisely two blocks for each  $Y_{ij}$  field. For instance, for  $Y_{43}$  we have  $\text{Tr } Y_{43} U_{35}^1 V_{54}^2$  and  $\text{Tr } Y_{43} V_{38}^2 U_{84}^1$ . Let us next consider the variable  $f_7$ . From (5.14) and the last equation in (5.3) we see that the charges of the corresponding operator are  $(-\frac{1}{2}, 5, 0)$ . We found two gauge invariant products of the fields with these quantum numbers:

$$f_7 \quad \longleftrightarrow \quad \text{Tr } U_{42}^1 Z_{21} U_{13}^1 V_{38}^2 U_{85}^2 V_{54}^1 \quad \text{and} \quad \text{Tr } U_{72}^2 Z_{21} U_{13}^1 V_{38}^2 U_{84}^1 V_{47}^2. \quad (5.15)$$

Let us show as a simple exercise that if one imposes the  $F$ -term conditions, the two operators above become equivalent. Indeed, from the  $F$ -term condition for the field  $Y_{25}$  it is clear that we can replace the  $V_{54}^2 U_{42}^1$  in the first sequence of the fields by  $V_{57}^1 U_{72}^2$ . Next, in order to arrive at the second operator in (5.15) we have to replace  $U_{85}^2 V_{57}^1$  by  $U_{84}^1 V_{47}^2$  using the  $F$ -term condition for the field  $Y_{78}$ . Using similar steps it can be shown that the last equation in (5.4) indeed holds when we replace the polynomials  $f_7$ ,  $b$  and  $d$  by the appropriate operators.

Let us represent the corresponding operators for the rest of the polynomials:

$$\begin{aligned} f_1 &\longleftrightarrow \text{Tr } Y_{43} U_{35}^1 V_{57}^1 U_{76}^1 V_{68}^1 U_{84}^1, \dots \\ f_2 &\longleftrightarrow \text{Tr } Y_{56} V_{63}^2 V_{38}^2 U_{85}^2, \dots \\ f_3 &\longleftrightarrow \text{Tr } Y_{43} U_{35}^1 Y_{56} V_{68}^1 U_{84}^1, \dots \\ f_4 &\longleftrightarrow \text{Tr } Y_{43} Y_{37} U_{72}^2 Y_{25} V_{54}^2, \dots \\ f_5 &\longleftrightarrow \text{Tr } U_{84}^1 V_{47}^2 U_{72}^2 Z_{21} U_{16}^2 V_{63}^2 V_{38}^2, \dots \\ f_6 &\longleftrightarrow \text{Tr } V_{38}^2 U_{84}^1 U_{42}^1 Z_{21} U_{13}^1, \dots \end{aligned} \quad (5.16)$$

Here the dots remind us that in general there are many other operators related to the same polynomial, which are equivalent by virtue of the  $F$ -term relations.

Finally, we are in a position to identify two non-holomorphic “short-cut” operators corresponding to the  $H = 1$  excitation of the string, as we have discussed in the end of the previous section.

Let us focus on the null geodesic that lies at  $x = x_2$  and  $\theta = \frac{\pi}{2}$ . We saw in the previous section that for chiral primary operators the string Hamiltonian takes the following form (see (4.9) and the discussion following it):

$$H_{\text{C.P.}} = n_2 + n_4. \quad (5.17)$$

This immediately implies that the polynomial corresponding to the ground state  $H = 0$  is  $a$ , which in turn is associated with the chiral operator  $\mathcal{L}_-^\dagger$ . Furthermore, the first excited state with  $(n_2, n_4) = (0, 1)$  is related to the polynomial  $c$  and the corresponding operator is  $\mathcal{L}_-^\dagger$ . From (5.1) it follows that for  $(n_2, n_4) = (1, 0)$  the polynomial is  $f_3$  and the relevant operator appears in (5.16).

Now let us address the construction of the “short-cut” operators. In the  $Y^{p,q}$  example we multiplied the ground state operator by one of the  $U(1)$  currents and then removed a chiral primary operator corresponding to  $H = 1$ . The “short-cut” operator produced this way is expected to correspond to  $H = 1$ , since for the current we have  $H = 2$ . We will adopt this way of construction also for the case at hand. The only novel feature in the  $L^{1,7,3}$  case is that we will start from a product of two operators corresponding to the ground state. This, of course, does not alter the final  $H = 1$  result for the “short-cut” operator. For the first operator we have in a schematic way:

$$U_{35}^1 Y_{56} Y_{64} Y_{43} \cdot Y_{78} U_{84}^1 Y_{43} Y_{37} \cdot \bar{U}_{58}^2 U_{85}^2 = U_{35}^1 \bar{U}_{58}^2 U_{84}^1 Y_{43} \cdot Y_{56} Y_{64} Y_{43} Y_{37} Y_{78} U_{85}^2. \quad (5.18)$$

Here the first two terms on the left hand side are different representations of the  $H = 0$  “long” operator  $\mathcal{L}_-^\dagger$  and the third term corresponds to the  $U(1)$  current. The last term on the right hand side is the  $\mathcal{L}_-^\dagger$  operator. We therefore conclude that the “short-cut” operator we are interested in is:

$$\mathcal{O}^{(1)} = \text{Tr } U_{35}^1 \bar{U}_{58}^2 U_{84}^1 Y_{43}. \quad (5.19)$$

Clearly there are many other “short-cut” operators with exactly the same quantum numbers, since we could have started with other representations for the operators  $\mathcal{L}_-^\dagger$  and  $\mathcal{L}_-^\dagger$ .

For the second operator we write:

$$U_{35}^1 Y_{56} Y_{64} Y_{43} \cdot Y_{37} Y_{78} U_{84}^1 Y_{43} \cdot \bar{V}_{86}^1 V_{68}^1 = \bar{V}_{86}^2 Y_{64} Y_{43} Y_{37} Y_{78} \cdot U_{35}^1 Y_{56} V_{68}^1 U_{84}^1 Y_{43}. \quad (5.20)$$

The first two terms on the left hand side are the “long” operator  $\mathcal{L}_-^\dagger$  and the third term corresponds to the  $U(1)$  current. On the right hand side the second term is an operator associated with the polynomial  $f_3$  (see (5.16)) and the first term is the the second “short-cut” operator:

$$\mathcal{O}^{(2)} = \text{Tr } \bar{V}_{86}^1 Y_{64} Y_{43} Y_{37} Y_{78}. \quad (5.21)$$

Again, there are many other operators equivalent to these “short-cut” operators that one can easily derive starting from different representations for the operators corresponding to the polynomials  $a$  and  $f_3$ .

## 6. Conclusions

In this paper we have investigated the Penrose limit of the  $Y^{p,q}$  and  $L^{p,q,r}$  families of Sasaki-Einstein geometries. The results presented here, therefore, extend the previous studies of [21, 22, 23]. In contrast to the Klebanov-Witten model, however, the quiver gauge theories dual to the new backgrounds are quite involved, so a straightforward analysis of the  $F$ -term relations becomes a formidable task. On the other hand working with polynomials of the Kähler quotient coordinates provides an easy way to identify the ground state BMN operator as well as the chiral operators dual to the first excited string state. We have also given a general idea how to construct non-chiral “short-cut” operators of [6] in the  $L^{p,q,r}$  models and provided an explicit solution in the  $L^{1,7,3}$  case.

Unfortunately, we were not able to perform the supergravity spectrum analysis in these backgrounds, in order to verify that the “short-cut” operators have proper dimensions matching the first excited string state. The first step towards this direction was done [37, 38] and it will be very interesting to pursue this direction in the future.

## Acknowledgments

We would like to thank Sergio Benvenuti, Leo Pando Zayas, Ami Hanany, Yaron Oz, Ricardo Argurio, Daniel Persson for fruitful discussions. The work of O. Mintkevich and J. Sonnenschein was partially supported by the Israel Science Foundation and by a grant of German-Israeli Project Cooperation - DIP Program (DIP H.52).

## References

- [1] J. P. Gauntlett, D. Martelli, J. Sparks, and D. Waldram, *Sasaki-Einstein metrics on  $S^2 \times S^3$* , *Adv. Theor. Math. Phys.* **8** (2004) 711–734, [[hep-th/0403002](#)].
- [2] D. Martelli and J. Sparks, *Toric geometry, Sasaki-Einstein manifolds and a new infinite class of  $AdS/CFT$  duals*, *Commun. Math. Phys.* **262** (2006) 51–89, [[hep-th/0411238](#)].
- [3] M. Bertolini, F. Bigazzi, and A. L. Cotrone, *New checks and subtleties for  $ads/cft$  and  $a$ -maximization*, *JHEP* **12** (2004) 024, [[hep-th/0411249](#)].
- [4] S. Benvenuti, S. Franco, A. Hanany, D. Martelli, and J. Sparks, *An infinite family of superconformal quiver gauge theories with Sasaki-Einstein duals*, *JHEP* **06** (2005) 064, [[hep-th/0411264](#)].
- [5] M. Cvetič, H. Lu, D. N. Page, and C. N. Pope, *New Einstein-Sasaki spaces in five and higher dimensions*, *Phys. Rev. Lett.* **95** (2005) 071101, [[hep-th/0504225](#)].
- [6] S. Benvenuti and M. Kruczenski, *Semiclassical strings in Sasaki-Einstein manifolds and long operators in  $N = 1$  gauge theories*, [hep-th/0505046](#).
- [7] S. Benvenuti and M. Kruczenski, *From Sasaki-Einstein spaces to quivers via BPS geodesics:  $L^{p,q,r}$* , [hep-th/0505206](#).
- [8] S. Franco *et. al.*, *Gauge theories from toric geometry and brane tilings*, *JHEP* **01** (2006) 128, [[hep-th/0505211](#)].

- [9] M. Cvetič, H. Lu, D. N. Page, and C. N. Pope, *New Einstein-Sasaki and Einstein spaces from Kerr-de Sitter*, **hep-th/0505223**.
- [10] A. Butti, D. Forcella, and A. Zaffaroni, *The dual superconformal theory for  $L^{p,q,r}$  manifolds*, *JHEP* **09** (2005) 018, [**hep-th/0505220**].
- [11] D. Martelli and J. Sparks, *Toric Sasaki-Einstein metrics on  $S^2 \times S^3$* , *Phys. Lett.* **B621** (2005) 208–212, [**hep-th/0505027**].
- [12] I. R. Klebanov and E. Witten, *Superconformal field theory on threebranes at a Calabi-Yau singularity*, *Nucl. Phys.* **B536** (1998) 199–218, [**hep-th/9807080**].
- [13] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, *Large  $N$  field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183–386, [**hep-th/9905111**].
- [14] D. Berenstein, J. M. Maldacena, and H. Nastase, *Strings in flat space and pp-waves from  $N=4$  super Yang Mills*, *JHEP* **04** (2002) 013, [**hep-th/0202021**].
- [15] R. Gueven, *Plane wave limits and  $t$ -duality*, *Phys. Lett.* **B482** (2000) 255–263, [**hep-th/0005061**].
- [16] M. Blau, J. Figueroa-O’Farrill, C. Hull, and G. Papadopoulos, *A new maximally supersymmetric background of IIB superstring theory*, *JHEP* **01** (2002) 047, [**hep-th/0110242**].
- [17] M. Blau, J. Figueroa-O’Farrill, and G. Papadopoulos, *Penrose limits, supergravity and brane dynamics*, *Class. Quant. Grav.* **19** (2002) 4753, [**hep-th/0202111**].
- [18] M. Blau, J. Figueroa-O’Farrill, C. Hull, and G. Papadopoulos, *Penrose limits and maximal supersymmetry*, *Class. Quant. Grav.* **19** (2002) L87–L95, [**hep-th/0201081**].
- [19] R. R. Metsaev, *Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background*, *Nucl. Phys.* **B625** (2002) 70–96, [**hep-th/0112044**].
- [20] R. R. Metsaev and A. A. Tseytlin, *Exactly solvable model of superstring in plane wave Ramond-Ramond background*, *Phys. Rev.* **D65** (2002) 126004, [**hep-th/0202109**].
- [21] N. Itzhaki, I. R. Klebanov, and S. Mukhi, *PP wave limit and enhanced supersymmetry in gauge theories*, *JHEP* **03** (2002) 048, [**hep-th/0202153**].
- [22] J. Gomis and H. Ooguri, *Penrose limit of  $N = 1$  gauge theories*, *Nucl. Phys.* **B635** (2002) 106–126, [**hep-th/0202157**].
- [23] L. A. Pando Zayas and J. Sonnenschein, *On Penrose limits and gauge theories*, *JHEP* **05** (2002) 010, [**hep-th/0202186**].
- [24] E. G. Gimon, L. A. Pando Zayas, J. Sonnenschein, and M. J. Strassler, *A soluble string theory of hadrons*, *JHEP* **05** (2003) 039, [**hep-th/0212061**].
- [25] I. R. Klebanov and M. J. Strassler, *Supergravity and a confining gauge theory: Duality cascades and chiral-resolution of naked singularities*, *JHEP* **08** (2000) 052, [**hep-th/0007191**].
- [26] J. M. Maldacena and C. Nunez, *Towards the large  $N$  limit of pure  $N=1$  super yang mills*, *Phys. Rev. Lett.* **86** (2001) 588–591, [**hep-th/0008001**].
- [27] S. Kuperstein and J. Sonnenschein, *Analytic non-supersymmetric background dual of a confining gauge theory and the corresponding plane wave theory of hadrons*, *JHEP* **02** (2004) 015, [**hep-th/0309011**].

- [28] R. Apreda, F. Bigazzi, and A. L. Cotrone, *Strings on pp-waves and hadrons in (softly broken)  $n = 1$  gauge theories*, *JHEP* **12** (2003) 042, [[hep-th/0307055](#)].
- [29] F. Bigazzi, A. L. Cotrone, and L. Martucci, *Semiclassical spinning strings and confining gauge theories*, *Nucl. Phys.* **B694** (2004) 3–34, [[hep-th/0403261](#)].
- [30] G. Bertoldi, F. Bigazzi, A. L. Cotrone, C. Nunez, and L. A. Pando Zayas, *On the universality class of certain string theory hadrons*, *Nucl. Phys.* **B700** (2004) 89–139, [[hep-th/0401031](#)].
- [31] F. Bigazzi, A. L. Cotrone, L. Martucci, and L. A. Pando Zayas, *Wilson loop, regge trajectory and hadron masses in a yang-mills theory from semiclassical strings*, *Phys. Rev.* **D71** (2005) 066002, [[hep-th/0409205](#)].
- [32] A. L. Cotrone, L. Martucci, J. M. Pons, and P. Talavera, *Heavy hadron spectra from spin chains and strings*, [hep-th/0604051](#).
- [33] S. S. Gubser and I. Mitra, *Some interesting violations of the Breitenlohner-Freedman bound*, *JHEP* **07** (2002) 044, [[hep-th/0108239](#)].
- [34] D. Berenstein, C. P. Herzog, P. Ouyang, and S. Pinansky, *Supersymmetry breaking from a Calabi-Yau singularity*, *JHEP* **09** (2005) 084, [[hep-th/0505029](#)].
- [35] S. Franco, A. Hanany, F. Saad, and A. M. Uranga, *Fractional branes and dynamical supersymmetry breaking*, *JHEP* **01** (2006) 011, [[hep-th/0505040](#)].
- [36] M. Bertolini, F. Bigazzi, and A. L. Cotrone, *Supersymmetry breaking at the end of a cascade of Seiberg dualities*, *Phys. Rev.* **D72** (2005) 061902, [[hep-th/0505055](#)].
- [37] H. Kihara, M. Sakaguchi, and Y. Yasui, *Scalar laplacian on Sasaki-Einstein manifolds  $Y^{p,q}$* , *Phys. Lett.* **B621** (2005) 288–294, [[hep-th/0505259](#)].
- [38] T. Oota and Y. Yasui, *Toric Sasaki-Einstein manifolds and Heun equations*, [hep-th/0512124](#).
- [39] S. Mukhi, M. Rangamani, and E. P. Verlinde, *Strings from quivers, membranes from moose*, *JHEP* **05** (2002) 023, [[hep-th/0204147](#)].
- [40] S. Kuperstein, *On Penrose limit of elliptic branes*, *JHEP* **06** (2003) 046, [[hep-th/0303042](#)].
- [41] D. Sadri and M. M. Sheikh-Jabbari, *Integrable spin chains on the conformal moose*, *JHEP* **03** (2006) 024, [[hep-th/0510189](#)].